Mixed-integer convex representability

IPCO 2017

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MICP

We consider mixed-integer convex programming (MICP). We define an MICP optimization problem as:

$$
\begin{aligned}
\min_{\boldsymbol{x}} \quad & \boldsymbol{c}^T \boldsymbol{x} : \\
& \boldsymbol{x} \in M, \\
& x_i \in \mathbb{Z}, \qquad\qquad \forall i \in I,
\end{aligned}
$$

where $M \subseteq \mathbb{R}^N$ is a closed, convex set, and some subset $I \subseteq [\![N]\!]$ of variables is constrained to take integer values. WLOG, the objective is linear, and $c_i = 0$ for $i \in I$.

MICP is a natural and useful generalization of **both** mixed-integer linear programming (MILP) **and** convex optimization.

Update from IPCO 2016

In IPCO 2016, L., Yamangil, Bent, & V. presented a paper on solving MICPs based on ideas from extended formulations (Tawarmalani and Sahinidis, Hijazi et al.) and disciplined convex programming.

Since then, our solver, `Pajarito`, has been rewritten from scratch with new algorithmic developments (Coey, L., & V).

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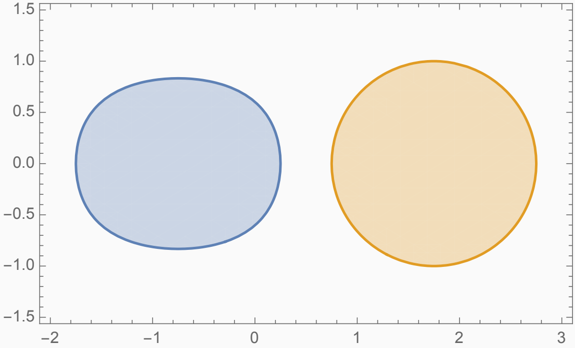
Now that we have a solver for MICP, what can we do with it? How broadly does it (and other MICP solvers) apply?

Known positive result



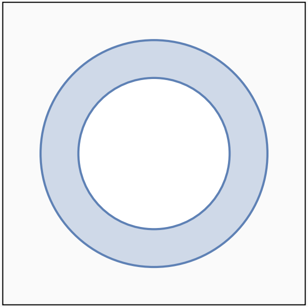
Ceria and Soares provide a construction of M such that after **projecting out the integer-constrained variables** we obtain (in particular) the set above. Hence we can use MICP to optimize over this nonconvex set.

The representability question

More generally, **which nonconvex sets can be obtained after projecting out the integer-constrained variables from MICP feasible regions?**

Specifically, if a nonconvex set S is MICP representable (projection of an MICP feasible region), then the nonconvex constraint $\boldsymbol{x} \in S$ can be enforced in an MICP problem. Finite intersections of these nonconvex constraints are also MICP representable.

For example, consider the annulus, a simple nonconvex set. Is it MICP representable?



More questions

Is the set of rank-1 matrices MICP representable?

What about the set $\{(a, b, c) : ab = c\}$ that could be used to model a product of variables?

Can we completely characterize MICP representable sets?

Definitions

Consider a set M in $\mathbb{R}^{n+p+d}$. Denote the variables in $\mathbb{R}^n$, $\mathbb{R}^p$, and $\mathbb{R}^d$ as $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$. Let:

$$\text{proj}_x(M) = \left\{\boldsymbol{x} \in \mathbb{R}^n : \exists (\boldsymbol{y}, \boldsymbol{z}) \in \mathbb{R}^{p+d} \text{ with } (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in M\right\}$$

We say M induces a formulation of S if,

$$S = \text{proj}_x\left(M \cap \left(\mathbb{R}^{n+p} \times \mathbb{Z}^d\right)\right).$$

The index set of the formulation is defined as,

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Notice,

$$S = \bigcup_{\boldsymbol{z} \in C \cap \mathbb{Z}^d} \text{proj}_x\left(M \cap (\mathbb{R}^{n+p} \times \{\boldsymbol{z}\})\right)$$

Definitions continued

- A set $S \subseteq \mathbb{R}^n$ is MILP representable if there exists a formulation of S with M polyhedral
- A set $S \subseteq \mathbb{R}^n$ is MICP representable if there exists a formulation of S with M closed and convex
- A set $S \subseteq \mathbb{R}^n$ is MINQP representable if there exists a formulation of S with M defined by (possibly nonconvex) quadratic constraints

Known results

MILP representability has been studied by Jeroslow and Lowe, Basu et al. (today), etc. If S is *rational* MILP representable, then there exist bounded rational polyhedra $P_1, \ldots, P_k$ and integer vectors $\boldsymbol{r}^1, \boldsymbol{r}^2, \ldots, \boldsymbol{r}^t$ such that:

$$S = \bigcup_{i=1}^{k} P_k + \operatorname{intcone}(\boldsymbol{r}^1, \boldsymbol{r}^2, \ldots, \boldsymbol{r}^t),$$

where $\operatorname{intcone}(\boldsymbol{z}^1, \boldsymbol{z}^2, \ldots, \boldsymbol{z}^k) = \{\sum_{i=1}^{k} \lambda_i \boldsymbol{z}^i : \boldsymbol{\lambda} \in \mathbb{Z}_+^k\}$.

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Very few results for general MICP representability; ellipsoidal case by Del Pia and Poskin and pure integer case by Dey and Morán impose conditions on the index set C.

Preview of results in this talk

We present results for three cases with different restrictions on the index set C.

- "Bounded" MICP: complete characterization
- General MICP: powerful necessary condition
- "Rational" MICP: complete characterizations for special classes of sets S

The bounded-MICP case

Definition

A set S is bounded-MICP (MILP) representable if there exists an MICP (MILP) formulation with an index set C which satisfies $\left|C \cap \mathbb{Z}^d\right| < \infty$. That is, there is a formulation with only finitely many feasible assignments of the integer variables $\boldsymbol{z}$ (e.g., $\boldsymbol{z} \in \{0, 1\}^d$).

Classical result: If S is bounded-MILP representable, then there exist bounded polyhedra $P_1, \ldots, P_k$ and a polyhedral cone R such that:

$$S = \bigcup_{i=1}^{k} P_k + R.$$

Bounded-MICP representability

Lemma

$S \subseteq \mathbb{R}^n$ is bounded-MICP representable if and only if there exist nonempty, closed, convex sets $T_1, T_2, \ldots, T_k \subset \mathbb{R}^{n+p}$ for some $p, k \in \mathbb{N}$ such that $S = \bigcup_{i=1}^{k} \operatorname{proj}_x T_i$.

This completely characterizes the bounded case, generalizing the result from Ceria and Soares which imposed a restriction on recession cones.

Proof.

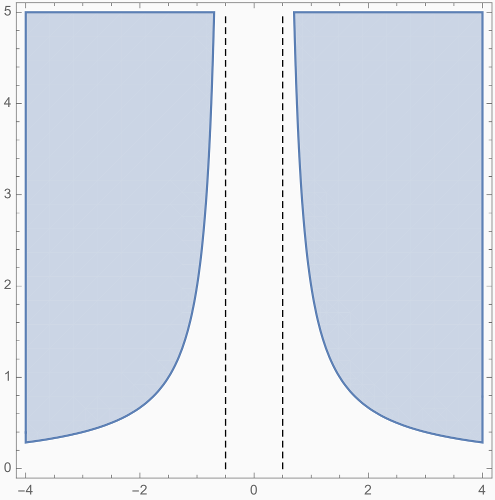
($\Leftarrow$) $\boldsymbol{x} \in S$ iff there exist $\boldsymbol{x}^i \in \mathbb{R}^n, \boldsymbol{y}^i \in \mathbb{R}^p$ for $i \in [\![K]\!]$ and $\boldsymbol{t} \in \mathbb{R}^k, \boldsymbol{z} \in \mathbb{Z}^k$ such that

$$\boldsymbol{x} = \sum_{i=1}^{k} \boldsymbol{x}^i, \quad (\boldsymbol{x}^i, \boldsymbol{y}^i, z_i) \in \hat{T}_i, \forall i \in [\![k]\!], \quad \sum_{i=1}^{k} z_i = 1, \; \boldsymbol{0} \leq \boldsymbol{z} \leq \boldsymbol{1},$$
$$||\boldsymbol{x}^i||_2^2 \leq z_i t_i, \forall i \in [\![k]\!], \boldsymbol{t} \geq 0,$$

where $\hat{T}_i$ is the closed conic hull of T_i, i.e., $\mathrm{cl}(\{(\boldsymbol{x}, \boldsymbol{y}, z) : (\boldsymbol{x}, \boldsymbol{y})/z \in T_i, z > 0\})$.

This defines a bounded-MICP representation of S. $\square$

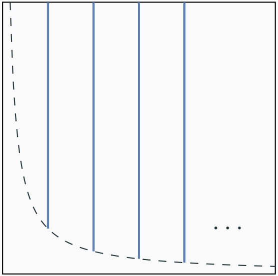
Bounded-MICP but not MILP



Region in blue is bounded-MICP representable. So is projection $(-\infty, -0.5) \cup (0.5, \infty)$.

The general case

We can get a **countably infinite** union of convex sets.



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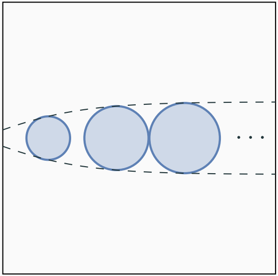

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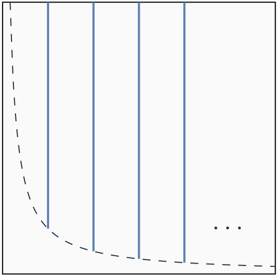

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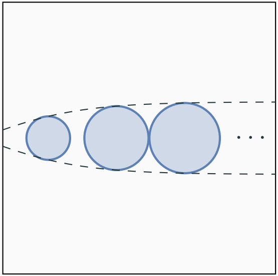

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Is it always a *countable* union? Yes, recall:

$$S = \text{proj}_x\left(M \cap \left(\mathbb{R}^{n+p} \times \mathbb{Z}^d\right)\right) = \bigcup_{\boldsymbol{z} \in C \cap \mathbb{Z}^d} \text{proj}_x\left(M \cap \left(\mathbb{R}^{n+p} \times \{\boldsymbol{z}\}\right)\right)$$

A simple necessary condition

Key idea: MICP-representable sets can be nonconvex, but not be "very" nonconvex!

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Definition

A set $S \subseteq \mathbb{R}^n$ is strongly nonconvex, if there exists a subset $R \subseteq S$ with $|R| = \infty$ such that for all distinct points $\boldsymbol{x}, \boldsymbol{y} \in R$,

$$\frac{\boldsymbol{x} + \boldsymbol{y}}{2} \notin S,$$

that is, an infinite subset of points in S such that the midpoint between any pair is not in S.

For example, a circle is a strongly nonconvex set but the union of finitely many convex sets and the integers $\mathbb{Z}^d$ for any $d \in \mathbb{N}$ are not!

A simple necessary condition

Lemma (The Midpoint Lemma)

Let $S \subseteq \mathbb{R}^n$. If S is strongly nonconvex, then S is not MICP representable.

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Corollary

The following sets are strongly nonconvex and therefore not MICP representable:

- *The annulus*
- *The set of $n \times n$ matrices of rank k for $k < n$*
- *The set $\{(a, b, c) \in \mathbb{R}^3 : ab = c\}$*
- *The graph of a nonlinear smooth function*
- *The set of prime numbers*
- *The set $\{1/n : n = 1, 2, \ldots\}$*

Proof of the midpoint lemma

Assume S is MICP representable. For some $M \subseteq \mathbb{R}^{n+p+k}$ which is closed and convex,

$$S = \mathrm{proj}_x \left(M \cap (\mathbb{R}^{n+p} \times \mathbb{Z}^d) \right).$$

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Each $\boldsymbol{x} \in R \subseteq S$ can be extended to $(\boldsymbol{x}, \boldsymbol{y}^x, \boldsymbol{z}^x) \in M$ for some $\boldsymbol{y}^x \in \mathbb{R}^p, \boldsymbol{z}^x \in \mathbb{Z}^d$.

Proof of the midpoint lemma

Notice that for any $\boldsymbol{x}^1, \boldsymbol{x}^2 \in R \subseteq S$ by convexity of M,

$$(\frac{\boldsymbol{x}^1 + \boldsymbol{x}^2}{2}, \frac{\boldsymbol{y}^{x^1} + \boldsymbol{y}^{x^2}}{2}, \frac{\boldsymbol{z}^{x^1} + \boldsymbol{z}^{x^2}}{2}) = \frac{1}{2}\left((\boldsymbol{x}^1, \boldsymbol{y}^{x^1}, \boldsymbol{z}^{x^1}) + (\boldsymbol{x}^2, \boldsymbol{y}^{x^2}, \boldsymbol{z}^{x^2})\right) \in M.$$

Hence, since $\frac{\boldsymbol{x}^1+\boldsymbol{x}^2}{2} \notin S = \mathrm{proj}_x\left(M \cap (\mathbb{R}^{n+p} \times \mathbb{Z}^d)\right)$, notice that this implies $\frac{\boldsymbol{z}^{x^1}+\boldsymbol{z}^{x^2}}{2} \notin \mathbb{Z}^d$.

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Therefore for any $\boldsymbol{x}^1, \boldsymbol{x}^2 \in R$ it necessarily holds $\frac{\boldsymbol{z}^{x^1}+\boldsymbol{z}^{x^2}}{2} \notin \mathbb{Z}^d$!

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Therefore for any $\boldsymbol{x}^1, \boldsymbol{x}^2 \in R$ it necessarily holds $\frac{\boldsymbol{z}^{x^1}+\boldsymbol{z}^{x^2}}{2} \notin \mathbb{Z}^d$!

But, by pigeonhole principle since the $(\boldsymbol{z}^x)_{\boldsymbol{x}\in R}$'s are infinitely many terms and the modulo 2 patterns in $\mathbb{Z}^d$ are finitely many, there exist two $\boldsymbol{x}^1, \boldsymbol{x}^2 \in R$ with $\boldsymbol{z}^{x^1} \equiv \boldsymbol{z}^{x^2}$ mod 2, so $\frac{\boldsymbol{z}^{x^1}+\boldsymbol{z}^{x^2}}{2} \in \mathbb{Z}^d$ a contradiction!

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Some form of rationality is essential: The set $S_0 := \left\{x \in \mathbb{N} : \sqrt{2}x - \lfloor \sqrt{2}x \rfloor \notin (\frac{1}{4}, \frac{3}{4})\right\}$ is (non-rational) MILP representable, which has a wild structure (infinite subset of the naturals but does not contain an arithmetic progression!)

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No! S_0 remains representable in this way!

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Definition

A set S is rational MICP representable if it has an MICP representation induced by the set M and the corresponding index set C is either bounded or rationally unbounded.

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Theorem

Let $S \subseteq \mathbb{N}$. Then S is rational MICP representable iff there exists a finite set S_0 and a rational-MILP-representable set S_1 such that $S = S_0 \cup S_1$.

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Comments:

- Our definition of rational MICP yields rigorous results
- Rational-MICP- and rational-MILP-representable subsets of the naturals are very similar

Some extensions to rational MICP

Post IPCO we have characterized rational MICP for two more classes.

- The family of piecewise linear functions defined on $\mathbb{R}$. Similar result as in the naturals: finite set of segments union an MILP-representable set!
- The family of bounded sets. Only union of finitely many compact convex sets is representable: no accumulation of any kind!

`https://arxiv.org/abs/1706.05135`

Summary

In this paper, we

- completely characterized the case when $C \cap \mathbb{Z}^d$ is finite in which case we get just a union of projections of closed convex sets
- studied the general case and found an easy necessary condition which lead to a number of negative results: low-rank, prime numbers, etc.
- introduced and analyzed in some cases rational MICP, an analogue of rational MILP.

Further research

This paper leads to some further research.

- Is the absence of strong nonconvexity also a sufficient condition for MICP-representability in some extent?
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Thanks! Questions?