Revealing Network Structure, Confidentially (Improved Rates for Node-Private Graphon Estimation)

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Introduction

Large and complicated networks arise everywhere in society! For example,

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- the disease transmission graph
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Analysis of Networks: Important across fields (sociology, medicine etc), rich in theory (random graphs, graph algorithms etc)

Privacy on Networks: Huge concern (e.g. Cambridge Analytica Scandal) and also rich in theory. Many open questions for networks!!

This work: Limits of Network Estimation under Privacy

New algorithms and **impossibility results**
for estimating complex network models,
subject to rigorous **privacy constraints** (**node differentially privacy**.)

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(1) **Stochastic Block Model-Estimation of probability matrix:**
-new analysis of recent private algorithm (BCS'15)
-matches in many regimes the **optimal non-private estimation rate**

(2) **Erdos-Renyi-Estimation of probability** p:
-Compute tightly **the optimal estimation rate**
-Uses a **novel extension lemma**, potentially of broad use

Node Differential Private Algorithms

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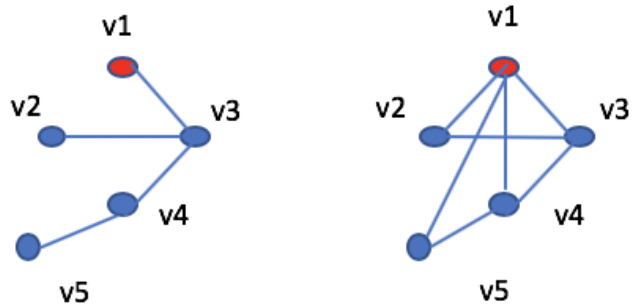

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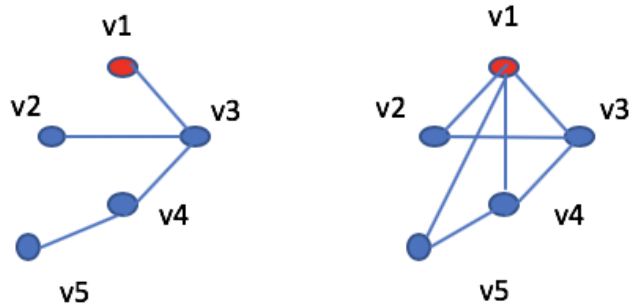

Definition

A randomized $\mathcal{A}$ on n-vertex graphs is ϵ-node-DP if for node-neighbors $G, G'$and S,

$$\exp(-\epsilon)\,\mathbb{P}\left(\mathcal{A}(G') \in S\right) \leq \mathbb{P}\left(\mathcal{A}(G) \in S\right) \leq \exp(\epsilon)\,\mathbb{P}\left(\mathcal{A}(G') \in S\right).$$

Modeling Large Networks: k-Stochastic Block Model

1-SBM (Erdos Renyi) $G(n, p)$: n nodes every edge appears independently with probability p.

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2-SBM $G(n, \begin{bmatrix} p_{1,1} & p_{1,2} \\ p_{2,1} & p_{2,2} \end{bmatrix})$ with $p_{1,2} = p_{2,1}$: n nodes, **2 groups** (each node chooses u.a.r.), and each edge between v_i, v_j **with probability** $p_{group(v_i),group(v_j)}$.

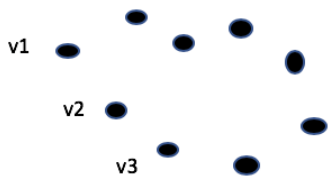

Figure: 9 vertices

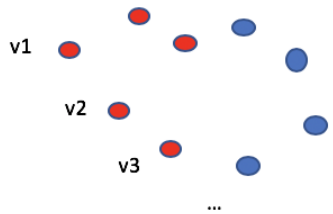

Figure: 2 groups

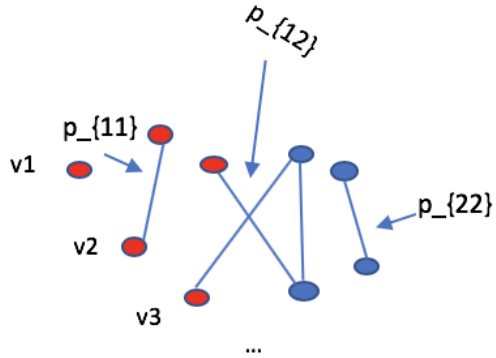

Figure: Assign Edges

Modeling Large Networks: k-Stochastic Block Model

k-SBM, $G(n, B)$, for sym. $B \in [0,1]^{k \times k}$:
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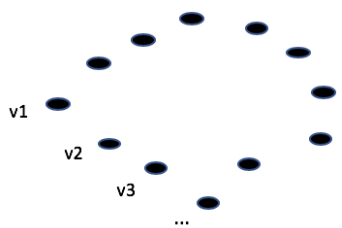


Figure: n = 12

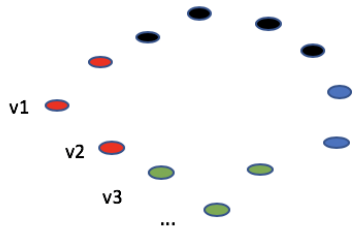
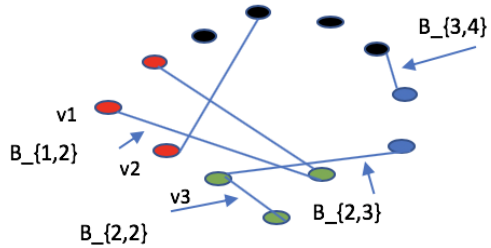


Figure: k = 4



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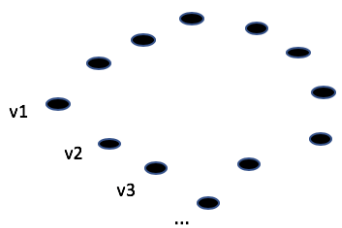
Constraint (!) : (ρ-**sparse**) **k-SBM**, $G(n, B)$, where $B \in [0, \rho]^{k \times k}$.

(Vast literature - planted bisection, planted clique, graph limits etc)



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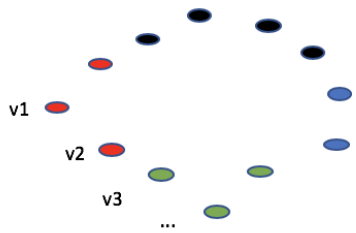

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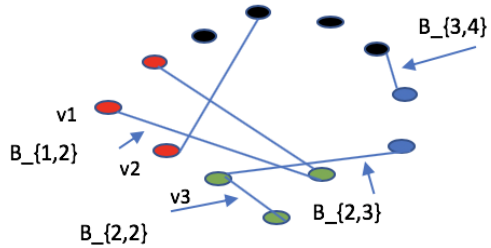

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$$R_k(\epsilon) = \min_{\mathcal{A}\ \epsilon\text{–node-DP}} \mathrm{err}(\mathcal{A}).$$

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(For agnostic learning see paper!)

k-SBM Upper Bound

Theorem (informal)

For any $\epsilon > 0$,

$$\mathcal{R}_k(\epsilon) = O\left(\rho\left(\frac{k^2}{n^2} + \frac{\log k}{n}\right)\right) + O\left(\rho^2 \frac{(k-1)^2 \log n}{n\epsilon} + \frac{1}{n^2\epsilon^2}\right).$$

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- *Intuition:* $\frac{k^2}{n^2}$ parametric rate for B, $\frac{\log k}{n} = \frac{\log k^n}{n^2}$ combinatorial rate
- Via a new detailed analysis of an ϵ-node-DP algorithm proposed in (BCS '15).

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Comments:

- (GLZ'14), (MS'17), (KTV'17): Optimal ϵ-independent part.
- Many regimes (e.g. ϵ, k constant and $\frac{1}{n} < \rho < \frac{1}{\log n}$):
  -(BCS'15) algorithm, optimal over **all** algorithms!
  -**No additional error** with privacy!

A lower bound for $k \geq 2$

Suppose each node $i \in [n]$ chooses the group in a close to uniform way. (Say each group has probability in $[\frac{1}{4k}, \frac{4}{k}]$.)

Proposition (informal)

For $k \geq 2$ *and any* $\epsilon > 0$,

$$\mathcal{R}_k^*(\epsilon) = \Omega\left(\frac{1}{n^2\epsilon^2}\right),$$

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Proof: reduction to privately estimating $q \in [0, 1]$ out of n samples from Bern(q).

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Upper bound by main result, Laplace noise to edge density, median of degrees etc.
Lower bounds, by vanilla methods such as packing arguments.

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What is the true ϵ-dependent rate?!

The case $k = 1$: $\frac{1}{n^4\epsilon^2} \leq \epsilon - \text{dep.} \leq \frac{1}{n^2\epsilon^2}$

Theorem

For $\epsilon > \frac{\log n}{n}$,

$$\mathcal{R}_1(\epsilon) = O(\frac{1}{n^2} + \frac{\log n}{n^3\epsilon^2}).$$

Many novel techniques including a general extension lemma (next slide)!

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Proposition (n^3 is tight!)

Furthermore, if G *is sampled u.a.r. from graphs with a fixed number of edges (conditional Erdos Renyi) for* ϵ *constant,*

$$\mathcal{R}_1'(\epsilon) = \Omega(\frac{1}{n^3\epsilon^2}).$$

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Technical challenge with *designing* differential private algorithms:

- **Privacy** constraint should hold for **any** pair of datasets
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Proposition ("Extending Private Algorithms at ϵ-cost")

Let $\hat{\mathcal{A}}$ ϵ-DP on a subset of the input space $\mathcal{H} \subseteq \mathcal{M}$. Then there exists $\mathcal{A}$ defined on $\mathcal{M}$ which is 1) 2ϵ-DP on $\mathcal{M}$ and 2) for every $\mathsf{D} \in \mathcal{H}$, $\mathcal{A}(\mathsf{D}) \stackrel{\mathsf{d}}{=} \hat{\mathcal{A}}(\mathsf{D})$.

Generalizes "extensions" from (KNRS'13), (BBDS'13), (CZ'13), (BCS'15), (RS'15).

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